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A POTENTIAL REDUCTION ALGORITHM FOR GENERALIZED LINEAR COMPLEMENTARITY PROBLEM OVER A POLYHEDRAL CONE

FENGMING MA

School of Operations Research and Management Sciences
Qufu Normal University
Rizhao Shandong, 276800
P. R. China
e-mail: rzhmafengming@163.com

Abstract

In this paper, we propose a potential reduction algorithm for generalized linear complementarity problem (GLCP) over a polyhedral cone. Under mild condition, we show that this algorithm can find an approximate solution of GLCP in finite iterations. Preliminary numerical experiments shows that the method has a good performance.

1. Introduction

The generalized linear complementarity problem (GLCP), is to find an $x^* \in R^n$ such that

$$F(x^*) \in \mathcal{K}, \quad G(x^*) \in \mathcal{K}^{\mathrm{o}}, \quad F(x^*)^{\top} G(x^*) = 0,$$

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where F and G are both affine mappings in R^n, and $\mathcal{K}$ is a polyhedral cone in R^n. That is, there exist matrices $P, Q \in R^{n\times n}$, $A \in R^{s\times n}$, $B \in R^{t\times n}$, and vectors $p, q \in R^n$ such that

$$F(x) = Px + p, \quad G(x) = Qx + q, \quad \mathcal{K} = \{v \in R^n \mid Av \geq 0,\ Bv = 0\}.$$

It is easy to verify that its polar cone $\mathcal{K}^{\mathrm{o}}$ has the following representation:

$$\mathcal{K}^{\mathrm{o}} = \{v \in R^n \mid v = A^{\top}\lambda + B^{\top}\mu,\ \lambda \geq 0,\ \lambda \in R^s,\ \mu \in R^t\}.$$

In this paper, we consider the case that $m = n$. For GLCP, Andreani et al. [1], reformulated it as a constrained optimization problem over R^n_+ and gave some conditions under which a stationary point of the optimization problem is a solution of GLCP. And Zhang et al. [8], reformulated it as an unconstrained optimization and gave a Newton-type algorithm which has global and quadratical convergence.

Motivated by Kojima et al. [3], we give a potential reduction algorithm which can start from an arbitrary interior feasible solution to obtain a solution in finite steps. The main contribution of this paper is threefold. Firstly, we discuss existence of solution of GLCP. Secondly, we give a condition under which Jacobian matrix of a system of equations in connection with GLCP is nonsingular, as well as some property of solution to Newton equation of the same system of equations. Finally, we describe detailed potential reduction algorithm and its convergence.

We end this section with some notations used in this paper. For vector $a \in R^n$, $D_a = \mathrm{diag}(a)$ denotes the diagonal matrix in which the i-th diagonal element is a_i. The inner product of vectors $x, y \in R^n$ is denoted by $x^{\top}y$. Let $\|\cdot\|$ and $\|\cdot\|_{\infty}$ denote the Euclidean norm and the infinity norm, respectively. The vector e denotes the vector of all ones, whose dimension is defined by the context of its use.

2. Solutions Existence and Nonsingularity Condition

In this section, we will first establish an equivalent reformulation of the GLCP, and then discuss the solution existence of the GLCP.

For the GLCP, the following conclusion is straightforward, also see [6].

Theorem 2.1. *A point* $x^* \in R^n$ *is a solution of the GLCP, if and only if there exist* $\lambda^* \in R^s$, $\mu^* \in R^t$ *such that*

$$A(Mx^* + p) \geq 0, \quad \lambda^* \geq 0, \quad (\lambda^*)^\top A(Mx^* + p) = 0,$$

$$B(Mx^* + p) = 0, \quad Nx^* + q = A^\top \lambda^* + B^\top \mu^*.$$

Define the vector-valued function $H : R^{n+2s+t} \to R^{n+2s+t}$ as

$$H(x, \lambda, \mu, \nu) := \begin{pmatrix} \lambda \circ \nu \\ A(Mx + p) - \nu \\ B(Mx + p) \\ Nx + q - A^\top \lambda - B^\top \mu \end{pmatrix},$$

where variables $x \in R^n$, $\lambda, \nu \in R^s$, $\mu \in R^t$, and $\lambda \circ \nu = (\lambda_1 \nu_1, \lambda_2 \nu_2, \cdots, \lambda_s \nu_s)^\top$. Then the GLCP can be equivalently reformulated as the following constrained equations:

$$H(x, \lambda, \mu, \nu) = 0 \quad \text{s.t.} \quad \lambda \geq 0, \nu \geq 0. \tag{2.1}$$

Based on this analysis, we can establish the main result in this section [3].

Theorem 2.2. *For the GLCP, if matrix* $N^\top M$ *is positive definite and set* S_+ *is nonempty, then the solution set of the GLCP, denoted by* S^*, *is nonempty, where*

$$S_+ = \left\{ (x, \lambda, \mu, \nu) \in R^{n+2s+t} \middle| \begin{array}{l} A(Mx + p) = \nu, \ B(Mx + p) = 0 \\ \\ Nx + q = A^\top \lambda + B^\top \mu, \ \lambda \geq 0, \ \nu \geq 0 \end{array} \right\}.$$

In our designed algorithm given in the next section, we need to solve a linear system of equations to obtain the searching direction at each direction, where the coefficient matrix of the linear system is the Jacobian of system $H(x, \lambda, \mu, \nu)$ defined in (2.1). To guarantee the efficiency of the algorithm, we give the following conditions under which the Jacobian of system $H(x, \lambda, \mu, \nu)$ is nonsingular. The proof of it is similar with that of Theorem 4.2 in [6].

Theorem 2.3. *For any* $x \in R^n$, $\lambda, \nu \in R^s_{++}$, $\mu \in R^t$, *if matrix* $N^\top M$ *is positive definite and* B *has a full row rank, then Jacobian matrix of* $H(x, \lambda, \mu, \nu)$ *is nonsingular.*

3. Analysis and Algorithm

It is well known that the potential reduction methods for solving the LP and the LCP are based on a carefully chosen potential function, and the concerned problem is solved by an iterative procedure, which makes the potential function value tend to $-\infty$ along with the progress of iteration. To extend the potential reduction method to the GLCP, we adopt the scaled Tanabe-Todd-Ye primal-dual potential function proposed in [4, 5]:

$$\Phi(z) = (s + \gamma) \log \lambda^\top \nu - \sum_{i=1}^{s} \log \lambda_i \nu_i - s \log s,$$

where $z = (x, \lambda, \mu, \nu)$, and $\gamma > 0$ is a parameter. It is easy to check that for $z \in S_{++} := \{z \in S_+ \mid \lambda > 0, \nu > 0\}$,

$$\Phi(z) \to -\infty, \text{ if and only if } z \to S^*.$$

Based on this analysis, we will establish a line search method to generate a sequence $\{z_k\} \subset S_{++}$ such that the sequence $\{\Phi(z_k)\}$ is strictly decreasing and diverges to $-\infty$.

First, we discuss the choice of the searching direction. For the current point $z \in S_{++}$, we use the following linear system of equations w.r.t. $d = (\Delta x, \Delta\lambda, \Delta\mu, \Delta\nu)$,

$$\nabla H \cdot d = (h, 0, 0, 0), \tag{3.1}$$

to generate a searching direction, where parameter $\beta \in [0, 1]$, and $h = -(D_\lambda \nu - \beta \frac{\lambda^\top \nu}{s} e)$. By Theorem 2.3, we know (3.1) has a unique solution.

We are now in a position to choose appropriate parameters β and $\theta > 0$ such that $\hat{z} = z + \theta d$ satisfying $\hat{z} \in S_{++}$ and $\Phi(\hat{z}) < \Phi(z)$. For this issue, we have the following result similar with [7].

Theorem 3.1. *If there exists* $\tau \in [0, 1)$ *such that*

$$\frac{\theta}{v_{\min}} \|V^{-1} h\| = \tau, \quad \tau \in [0, 1),$$

then $\hat{z} \in S_{++}$ *and*

$$\Phi(\hat{z}) - \Phi(z) \le -\delta,$$

where $\delta = \frac{1}{2}\sigma_1 \bar{\tau}, \bar{\tau} = \min\{\frac{1}{3}, \frac{\sigma_1}{4\sigma_2}\}, \sigma_1 = \min\{\frac{1}{2}, \frac{\gamma}{\sqrt{2s}}\}, \sigma_2 = \max\{1, \frac{\gamma}{4s}\}$.

Based on the information above, we now give the specific description of our potential reduction algorithm for solving GLCP.

Algorithm 3.1

Step 1. Given $\varepsilon > 0$. Choose initial point $z^0 = (x^0, \lambda^0, \mu^0, \nu^0) \in S_{++}$, parameters $\beta = \frac{s}{s+\gamma}, \delta = \frac{1}{2}\sigma_1 \bar{\tau}$, where $\bar{\tau} = \min\{\frac{1}{3}, \frac{\sigma_1}{4\sigma_2}\}, \sigma_1 = \min\{\frac{1}{2}, \frac{\gamma}{\sqrt{2s}}\}, \sigma_2 = \max\{1, \frac{\gamma}{4s}\}$. Let $k := 0$.

Step 2. If $(\lambda^k)^\top \nu^k \le \varepsilon$, then stop. Otherwise, let $z = z^k$.

Step 3. Solve (3.1) to get search direction d, and let $\hat{z} = z + \theta d$, where $\theta = \bar{\tau}\frac{v_{\min}}{\|V^{-1}h\|}$, $h = -(D_\lambda \nu - \beta \frac{\lambda^\top \nu}{s} e)$.

Step 4. Let $z^{k+1} = \hat{z}$, $k := k + 1$, and go to Step 2.

From the description of potential reduction algorithm, we can easily see that Algorithm 3.1 can find an ε-solution after at most $O(\sqrt{s} \cdot \log \frac{2^{\Phi(z^0)/\gamma}}{\varepsilon})$ iterations, i.e., our algorithm has polynomial complexity.

4. Numerical Results

In this section, we present some numerical experiments for our algorithm. Throughout our computations, we take parameters $\varepsilon = 10^{-14}$. The notation Iter denotes the number of iterations. All numerical results are reported for our examples with different dimension n and parameter γ.

Example 4.1. This problem is a GLCP over R^4_+, in which $F(x) = Px + p$, $G(x) = Qx + q$, where

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

$$p = q = (1, 1, 1, 1)^\top.$$

In Table 1, our results of this problem are reported with different γ and initial points.

Table 1. Numerical results for Example 4.1

γ	$\gamma = 2$			$\gamma = 8$		
x^0	$(0, 0, 0, 0)^\top$	$(1, 1, 1, 1)^\top$	$(2, 2, 2, 2)^\top$	$(0, 0, 0, 0)^\top$	$(1, 1, 1, 1)^\top$	$(2, 2, 2, 2)^\top$
Iter	239	298	325	275	334	360

γ	$\gamma = 16$		
x^0	$(0, 0, 0, 0)^\top$	$(1, 1, 1, 1)^\top$	$(2, 2, 2, 2)^\top$
Iter	311	370	392

Example 4.2. This example is LCP used by Geiger and Kanzow [2]. Let $F(x) = Mx + q$, where

$$M = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -1 & 4 \end{pmatrix}, \quad q = (-1, -1, \cdots, -1)^\top.$$

For this problem, we take the initial point $x^0 = (1, 1, \cdots, 1)^\top$. Table 2 lists the numerical results of this problem with different dimension n and parameter γ.

Table 2. Numerical results for Example 4.2

dimension	$n = 2$			$n = 4$			$n = 16$		
γ	$\gamma = 1$	$\gamma = 4$	$\gamma = 6$	$\gamma = 2$	$\gamma = 8$	$\gamma = 32$	$\gamma = 4$	$\gamma = 16$	$\gamma = 128$
Iter	144	144	305	254	274	620	586	628	1609

5. Conclusion

In this paper, we have analyzed the condition under which GLCP has solution and some property of solution of GLCP. We have also proposed a potential reduction algorithm for GLCP over a polyhedral cone. This algorithm may find an approximate solution of GLCP in finite steps by reducing value of potential function, not of $\lambda^{\top}\nu$. The primary numerical experiments proved our algorithm is effective.

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